Preprint typeset in JHEP style. - HYPER VERSION

NMCPP 2002/09

Riemannian Gauge Theory and Charge Quantization

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Abstract: Particle theorists build gauge theories around an action with a symmetry group. The gauge field of the action is similar to the connection coefficient in the Riemannian covariant derivative. The field strength tensor is similar to the curvature tensor. Inspired by these parallels with Riemannian geometry, we propose a different construction. Instead of defining a gauge theory by a group, we define a gauge theory by a vector bundle with fiber $F = \mathbf{R}^n$ or $F = \mathbf{C}^n$. Matter fields are Lorentz-invariant n-vectors on the vector-space fiber. We set up orthonormal gauge basis vectors that span the vector bundle. By expressing gauge-covariant fields in terms of orthonormal gauge basis vectors, we obtain an $SO(n)$ or $U(n)$ gauge-covariant derivative. All matter fields on a particular fiber couple with the same coupling constant. Even the matter fields on a $\mathbf{C}^1$ fiber, which have a $U(1)$ symmetry group, couple with the same charge of $\pm q$. In this geometrical gauge theory, the multiple standard-model $U(1)$ hypercharges require unnatural constraints among the curvatures of independent vector bundles. These constraints are naturally supplied by theories with grand unification. Because our action is independent of the choice of basis, its natural invariance group is $GL(n, \mathbf{R})$ or $GL(n, \mathbf{C})$.

Keywords: Gauge Theory, Riemannian Geometry, Vector Bundles, Embedding, Charge Quantization.

Contents

1. Gauge Theory

Particle theorists have developed a traditional way to construct a gauge theory. They define a continuous local symmetry group G and a transformation rule $\phi'(x) = g(x)\phi(x)$ for every multiplet $\phi(x)$ of matter fields, where $g(x)$ is some representation of G. They maintain the gauge invariance of the matter-field derivative by introducing a gauge field A_μ and a gauge-field transformation rule

$$A'_\mu = g\, A_\mu\, g^{-1} + (i/q)\, g\, \partial_\mu\, g^{-1} \tag{1.1}$$

that cancels the extra term appearing in the derivative $\partial_\mu(g\,\phi)$. In the action they include every Lorentz-invariant, gauge-invariant, renormalizable interaction.

The traditional gauge-theory Lagrangian has many parallels with Riemannian geometry [1, 2, 3]. The matter-field derivatives appear together with the gauge fields,

$D_\mu\phi = (\partial_\mu - iqA_\mu)\phi$, in a structure that resembles the covariant derivatives of Riemannian geometry ∇_μ. The field-strength tensor $F_{\mu\nu}$, which is constructed to maintain the gauge invariance of $\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$, is the commutator of two covariant derivatives

$$-i\, F_{\mu\nu}{}^a{}_b \phi^b = ([D_\mu, D_\nu]\phi)^a \tag{1.2}$$

as is the curvature tensor of Riemannian geometry

$$R_{\mu\nu}{}^\lambda{}_\sigma V^\sigma = ([\nabla_\mu, \nabla_\nu]V)^\lambda. \tag{1.3}$$

A natural description of nature should treat the four known forces with some degree of symmetry. General relativity is based upon Riemannian geometry. What changes occur if a gauge theory is drawn from Riemannian geometry rather than defined by gauge invariance? We offer some conclusions in what follows. In section 2 we review charge quantization in traditional gauge theory. Section 3 is about Riemannian geometry. In section 4 we describe a gauge theory inspired by Riemannian geometry. Section 6 discusses charge quantization and symmetry groups in this geometrical gauge theory. Section 7 summarizes our conclusions.

2. Charge Quantization

Traditional gauge theories, as described above, naturally quantize the charges of non-abelian gauge groups but not those of abelian groups.

In the abelian case, each field ϕ_i can be defined with a transformation law specific to its own coupling constant q_i. The abelian field-strength tensor (1.2) is a linear function of the coupling constant, $F(q_1 + q_2, A) = F(q_1, A) + F(q_2, A)$, and of the gauge fields, $F(q, A_1 + A_2) = F(q, A_1) + F(q, A_2)$. The field-strength term $\alpha\, F_{\mu\nu}F^{\mu\nu}$ in the action can be multiplied by any constant α. Because of the field-strength tensor's linearity and the arbitrariness of α, the covariant derivative of any field ϕ_i can be used in Eq.(1.2) to define $F_{\mu\nu}$. The field strength tensor is strictly gauge invariant regardless of which coupling constant one chooses in the gauge-field transformation rule (1.1).

In contrast, a non-abelian gauge group only allows one coupling constant q for all non-trivially transforming fields. The non-abelian field-strength tensor is not linear in the coupling constant, nor is it linear in the gauge field. If different fields had different coupling constants, then one would have to choose which coupling constant to use in the gauge-field transformation rule (1.1) of the gauge fields in the field-strength tensor. Also, one would need to transform the gauge fields in the field-strength tensor differently from the gauge fields in the covariant derivatives of the matter fields. This difference would violate gauge invariance. So all fields transforming under a non-abelian gauge group must have the same coupling constant.

The coupling constant is different from the charge. In the non-abelian case, one will have different charges for matter-field multiplets proportional to different eigenvectors of the group generators. Simple groups will have quantized charges. In contrast, an abelian group has a continuum of eigenvalues, and the matter fields may be proportional to any corresponding eigenvector. In the abelian case, no mathematical difference distinguishes fields coupling with different coupling constants from fields coupling with the same coupling constant but with different charges.

As far as we know, matter fields with an abelian gauge group have charges that are integral multiples of $q_e/3$ where q_e is the free-space charge of the electron. The electric-charge quantization problem is the conflict between these quantized electric charges and the unconstrained continuum of allowable electric charges.

Several physicists have offered explanations for the quantization of electric charge. Dirac [4] in 1931 showed that if magnetic monopoles exist, they would quantize electric charge. Georgi and Glashow [5] and Pati and Salam [6] explained the charges of the fermions by associating the photon with a traceless generator of an $SU(n)$ unification group. Others [7, 8, 9, 10, 11, 12] have exploited anomaly cancellation.

3. Riemannian Geometry

In Riemannian geometry [13, 14], a projection $\pi : E \to M$ from a total space E to a real, d-dimensional base manifold M gives rise to a fiber bundle that is the collection of vector spaces $\pi^{-1}(x) = \mathbf{R}^d$ for every space-time point $x \in M$. Because the fiber $\pi^{-1}(x)$ is a vector space, the fiber bundle is also a vector bundle.

The fiber $\pi^{-1}(x)$ becomes a tangent space at the point p when one identifies directions and lengths on the fiber with directions and lengths on the base manifold. Since the tangent fiber is a flat tangent plane, only one set of basis vectors $t_\mu(x)$ at a fixed origin $O \in \pi^{-1}(x)$ is needed to fully define coordinates on the tangent space.

A vector field V is a tangent-space vector that varies smoothly with the point p in M. In a particular basis t, the vector field is given by $V = V^\mu t_\mu$ where the Greek indices run through the dimensionality of the manifold. The functions V^μ are real-valued coefficients (i.e., not vectors) of the basis vectors t_μ. A common choice for t_μ is the coordinate tangent vectors, $t_\mu = \partial_\mu$.

In addition to basis vectors, we need dual basis vectors and an inner product. The dual basis vector t^μ is defined as the linear operator on the basis vectors that returns the Kronecker delta: $t^\mu(t_\nu) = \delta^\mu_\nu$. The dual basis vectors combine with the basis vectors to form a projection operator $P = t_\mu t^\mu$ which projects vectors onto the tangent space. For a vector V in the tangent space, the projection operator gives simply, $P(V) = t_\mu t^\mu(V^\alpha t_\alpha) = t_\mu V^\alpha \delta^\mu_\alpha = t_\mu V^\mu = V$. The inner product $\langle V, W\rangle$ measures the length or the point-wise overlap of two vector fields; the inner product of two tangent vectors is the metric, $g_{\mu\nu} = \langle t_\mu, t_\nu \rangle$. In a particular basis, the inner product of two vectors is $\langle V, W\rangle = \langle V^\mu t_\mu, W^\nu t_\nu\rangle = V^\mu W^\nu \langle t_\mu, t_\nu\rangle = V^\mu W^\nu g_{\mu\nu}$.

The covariant derivative of a vector field V in the direction t_σ is the projection onto the tangent space of the derivative of the vector field V in the direction t_σ:

$$\nabla_\sigma(V) \equiv P(\partial_\sigma(V^\alpha t_\alpha)) \equiv t_\alpha(\delta^\alpha_\mu \partial_\sigma + \Gamma^\alpha_{\sigma\mu})V^\mu \tag{3.1}$$

$$\nabla_\sigma(V) = P(t_\alpha \partial_\sigma V^\alpha + V^\alpha \partial_\sigma t_\alpha) = t_\alpha \left(\delta^\alpha_\mu \partial_\sigma + t^\alpha(\partial_\sigma t_\mu)\right) V^\mu. \tag{3.2}$$

The Christoffel symbols are

$$\Gamma^\alpha_{\sigma\mu} \equiv t^\alpha(\partial_\sigma t_\mu). \tag{3.3}$$

This derivative of basis vectors involves a comparison of vectors at neighboring points on the manifold; it may be computed indirectly through derivatives of the metric, $\Gamma^\alpha_{\sigma\nu} = \frac{1}{2} g^{\alpha\lambda}(\partial_\sigma g_{\lambda\nu} + \partial_\nu g_{\sigma\lambda} - \partial_\lambda g_{\sigma\nu})$.

Another way to compare basis vectors at different points is to embed the manifold M in a Euclidean (or Lorentzian) space $\tilde{M}$ of higher dimension, $X : M \to \tilde{M}$. Vectors in $\tilde{M}$ are compared by means of ordinary parallel transport. Nash [15] proved that an isometric embedding X is possible with $\tilde{M} = \mathbf{R}^D$ as long as $D \geq d(3d + 11)/2$. The explicit tangent vectors are found by coordinate derivatives on the coordinates of the embedding function $\tilde{t}_\mu^{\,j} = \frac{\partial}{\partial x^\mu} X^j(x)$ yielding a $d \times D$-dimensional rectangular matrix. The metric on M is given by the inner product of two basis vectors $\langle t_\mu, t_\nu\rangle = g_{\mu\nu} = \delta_{ij}\tilde{t}^i_\mu \tilde{t}^j_\nu$. This explicit embedding aids understanding and visualization.

In Riemannian geometry, tangent-space basis vectors can completely describe the intrinsic properties of the manifold such as the connection, the metric, and the curvature tensors. They can also describe quantities that are not inherent to the manifold, such as the vector fields.

4. Riemannian Gauge Theory

We start from a fiber that is the vector space associated with the properties of a particular matter field, rather than from a gauge group and a symmetry transformation as in traditional gauge theory. Starting from this vector bundle, natural $SO(n)$ and $U(n)$ gauge groups follow from the universal connection of Narasimhan and Ramanan [16, 17]. With an additional constraint, so do $SU(n)$ gauge theories.

Atiyah [18]; Dubois-Violette and Georgelin [19]; Corrigan Fairlie, Templeton, and Goddard [20]; Cahill and Raghavan [21] all have used the Riemannian gauge theory described in this section to represent traditional gauge theory, as more recently have others [22, 23]. Instead of describing a traditional gauge theory with a geometrical model, we build a new gauge theory beginning with a geometrical description that closely parallels Riemannian geometry. Our approach distinguishes itself in several ways from traditional gauge theory.

Our Riemannian gauge theory begins with a vector bundle given by a projection $\pi_G : E_G \to M$ from a total space E_G to a space-time base manifold M. The gauge vector bundle is the collection of vector spaces $\pi_G^{-1}(x) = F|_x$ for every point $x \in M$. The fundamental matter field determines the choice of the vector space $F|_x$. For example, if ϕ is a real, n-dimensional matter-field multiplet, then the fiber is $F|_x = \mathbf{R^n}$; and if ϕ is a complex, n-dimensional field, then the fiber is $F|_x = \mathbf{C^n}$. We shall refer to the vector space $F|_x$ as the *fiber* or the *gauge fiber* in what follows.

The geometry of the gauge vector bundle differs from the tangent bundle because directions or lengths on the fiber $F|_x$ are not identified with directions or lengths on the base manifold. Like the tangent fiber, the gauge fiber is a flat plane, and so only one basis-vector set $e_a(x)$ at a fixed origin $O \in F|_x$ is needed to fully define its coordinates.

As in Riemannian geometry, a matter field $\phi(x) \in F_p$ varies smoothly with the point $x \in M$. In terms of a local basis $e_a(x)$ on $F|_x$, the matter field is given by $\phi(x) = \phi^a(x) e_a(x)$ where the Latin indices run through the dimensionality of the gauge fiber. The component $\phi^a(x)$ of the matter field $\phi(x)$ is a coefficient of a basis vector $e_a(x)$ and is not itself a vector. The complex conjugate of the component ϕ^a is denoted $\phi^{\bar{a}} \equiv \overline{\phi^a}$, similarly $e_{\bar{a}} \equiv \overline{e_a}$. The components $\phi^{\bar{a}}$ are the coefficients of the complex-conjugate basis vectors, as in $\overline{\phi} = \phi^{\bar{a}} e_{\bar{a}}$.

In addition to basis vectors, we need dual basis vectors and an inner product. Like the dual basis vectors of Riemannian geometry, the dual basis vector e^a is defined as the linear operator on the basis vectors that returns the Kronecker delta: $e^a(e_b) = \delta^a_b$. The dual basis vectors combine with the basis vectors to form a projection operator $P = e_a e^a$ which projects vectors onto the gauge fiber. For a matter field ϕ in the gauge fiber, the projection operator gives simply, $P(\phi) = e_a e^a(\phi^c e_c) = e_a \phi^c \delta^a_c = e_c \phi^c = \phi$. A complex vector space also has dual vectors for the complex conjugate of the basis vectors $e^{\bar{a}}(e_{\bar{b}}) = \delta^{\bar{a}}_{\bar{b}}$. By definition [14, p. 275], one has $e^{\bar{a}}(e_b) = 0$ and $e^a(e_{\bar{b}}) = 0$.

We use the notation $\langle \phi, \psi \rangle$ to denote the inner product of two matter vector fields, ϕ and ψ. In quantum mechanics inner products are used to compute lengths and probability amplitudes. The inner product of the basis vectors of the fiber $F|_x$ is the gauge-fiber metric $g_{\bar{a}b} = \langle e_a, e_b \rangle$, which is distinguished by its Latin indices from the metric $g_{\mu\nu}$ of the base manifold M. The inner product of two complex matter vectors is $\langle \phi, \psi \rangle = \langle \phi^a e_a, \psi^b e_b \rangle = \phi^{\bar{a}} \psi^b g_{\bar{a}b}$. This inner product uses a hermitian metric, $\overline{g_{\bar{a}b}} = g_{\bar{b}a}$, which by definition also satisfies $g_{\bar{a}\bar{b}} = g_{ab} = 0$. The gauge-fiber metric is defined so that $g_{\bar{a}b} = g_{b\bar{a}}$ and $\overline{g_{\bar{a}b}} = g_{a\bar{b}}$. The quantity $g^{a\bar{b}}$ is the inverse of the fiber metric, $g^{a\bar{b}} g_{\bar{b}c} = \delta^a_c$.

The fiber metric and its inverse can raise and lower indices:

$$\phi_{\bar{a}} = g_{\bar{a}b}\phi^b \qquad \phi^{\bar{a}} = g^{\bar{a}b}\phi_b. \tag{4.1}$$

Because $g^{ab} = 0$, we cannot get ϕ^a by using g^{ab} to raise an index. Instead we write $\phi^a = g^{a\bar{b}}\phi_{\bar{b}}$.

Like the covariant derivative in Riemannian geometry, the gauge-covariant derivative of a matter multiplet ϕ in the direction t_σ is the projection onto the gauge fiber of the derivative of the matter multiplet ϕ in the direction t_σ:

$$D_\sigma(\phi) \equiv P(\partial_\sigma(e_a\phi^a)) \equiv e_a \left(\delta^a_b\partial_\sigma - i(A_\mu)^a{}_b\right)\phi^b \tag{4.2}$$

$$D_\sigma(\phi) = P(e_a\partial_\sigma\phi^a + \phi^a\partial_\sigma e_a) = e_a\left(\delta^a_b\partial_\sigma + e^a(\partial_\sigma e_b)\right)\phi^b. \tag{4.3}$$

The gauge field is

$$(A_\mu)^a{}_b = ie^a(\partial_\mu e_b). \tag{4.4}$$

This derivative of basis vectors raises the question of how to compare them at neighboring points on the manifold. The traditional approach [24, p. 7] is to take the matter fields and the gauge fields as the fundamental objects of the theory with $(A_\mu)^a{}_b = (T^c)^a{}_b A^c_\mu(x)$, where the T^c are the generators of the gauge group.

An alternative way to compare basis vectors at different points of the manifold is to embed the gauge fiber F in a trivial, real or complex Euclidean vector bundle $M \times \tilde{F}$ with a fiber $\tilde{F}$ of higher dimension: $F \to \tilde{F}$. In the spirit of Nash's embedding theorem [15], Narasimhan and Ramanan [16, 17] showed that for any $U(n)$ or $SO(n)$ gauge field, one can embed the gauge fiber F in a trivial embedding fiber $\tilde{F}$ of dimension $N \geq (2\,d+1)n^3$. The basis vectors of the fiber $F \subset \tilde{F}$ now are the orthonormal vectors $\tilde{e}_a{}^j$, where the index j runs from 1 to N. The n basis vectors e_a of F are each N-vectors $\tilde{e}_a{}^j$ in the trivial fiber and span an n-dimensional subspace of $\tilde{F}$. The embedding space may now be used to express the projection operator

$$P^j{}_k = \tilde{e}_a{}^j\,\tilde{e}^a{}_k \tag{4.5}$$

and the metric

$$g_{\bar{a}b} = \overline{\tilde{e}_a{}^k}\,\tilde{e}_b{}^j\,\delta_{\bar{k}j} = \tilde{e}_{\bar{a}j}\,\tilde{e}_b{}^j = \sum_{j=1}^{N}\overline{\tilde{e}_{aj}}\,\tilde{e}_{bj}, \tag{4.6}$$

in which the bar denotes ordinary complex conjugation. The quantity $\delta_{\bar{j}k}$ may be interpreted as a hermitian metric on a complex Euclidean embedding fiber.

Narasimhan and Ramanan based their embedding theorem on orthonormal basis vectors $\langle e_a, e_b\rangle = \delta_{\bar{j}k}\,\overline{\tilde{e}_a{}^j}\,\tilde{e}_b{}^k = \delta_{\bar{a}b}$ spanning an n-dimensional subspace of an N-dimensional embedding space. Any two choices of real (complex) orthonormal basis vectors may be related by an orthogonal (unitary) transformation. This arbitrariness in the choice of basis vectors is gauge invariance. For real or complex orthonormal basis vectors, they showed that the resulting connection and matter fields have an $SO(n)$ or $U(n)$ gauge symmetry. The somewhat artificial constraint $\tilde{e}^a{}_j \partial_\mu \tilde{e}_a{}^j = 0$ leads to an $SU(n)$ gauge group.

The embedding $\tilde{e} : F \to \tilde{F}$ is not unique, and so we do not provide a general map $(A_\mu)^a{}_b \to \tilde{e}_a{}^j$ from the gauge field to the embedded basis vectors. The inverse map (4.4) from the embedded basis vectors to the gauge field is

$$(A_\mu)^a{}_b = i\tilde{e}^a{}_j \partial_\mu \tilde{e}_b{}^j . \tag{4.7}$$

Narasimhan and Ramanan have shown that this formula for the gauge field $(A_\mu)^a{}_b$ in terms of embedded orthonormal basis vectors is possible for every $SO(n)$, $U(n)$, or $SU(n)$ gauge field. The purpose of the embedding is to clarify the geometry and mathematics of the basis vectors.

In Riemannian gauge theory, the basis vectors of the gauge fiber can completely describe the intrinsic properties of the manifold such as the gauge field or connection A_μ and the curvature tensor $F_{\mu\nu}$ of the gauge fiber. They also can describe quantities that are not inherent to the gauge fiber geometry, such as the matter fields.

Several physicists [16, 17, 18, 19, 20, 21, 22, 23] have described traditional gauge theory in terms the orthonormal basis vectors of a gauge fiber. Here we instead consider what changes arise when we start from Riemannian gauge theory.

5. The Action

We construct the action for Riemannian gauge theory to be as similar as possible to the action of general relativity. We focus on writing the action in terms of inner products on the gauge fiber, which is equivalent to contracting upper indices with lower indices.

Both in general relativity and in Riemannian gauge theory, the action of the matter field is formed by inner products on the gauge fiber integrated over space-time, here taken to be four dimensional. For spinless bosons the action is

$$S_\phi = \int d^4x \sqrt{-g} \; \left[\langle D_\mu \phi, D^\mu \phi \rangle - \langle m\phi, m\phi \rangle \right], \tag{5.1}$$

and for fermions it is

$$S_\psi = \int d^4x \sqrt{-g} \; \left[\langle \psi, i\,\gamma^0\,\gamma^\mu\, D_\mu \psi \rangle + \langle \psi, m\,i\,\gamma^0 \psi \rangle \right]. \tag{5.2}$$

Both in general relativity and in Riemannian gauge theory, the action of the gauge fields measures the intrinsic Riemannian curvature. In general relativity, the curvature tensor (1.3) represents the change in a vector V due to parallel transport around a loop. If the loop lies in the t_μ-t_ν plane, this change is $(V - V')^\sigma = dx^\mu\, dx^\nu\, R_{\mu\nu}{}^\sigma{}_\lambda\, V^\lambda$. The tensor R is the fundamental measure of intrinsic curvature. The action of general relativity $\int d^4x\sqrt{-g}\, R_{\mu\nu\sigma\lambda}\, g^{\mu\lambda}\, g^{\nu\sigma}$ is formed from the contracted curvature tensor. The factor $\sqrt{-g}$ is the square root of the determinant of the metric of the base manifold.

In Riemannian gauge theory, the field-strength tensor (1.2) also represents the change $(\phi - \phi')^a = -i\, dx^\mu\, dx^\nu\, F_{\mu\nu}{}^a{}_b \phi^b$ in a vector V due to parallel transport around a loop. Therefore $F_{\mu\nu}{}^a{}_b$ measures the intrinsic curvature of the gauge fiber.

Both R and F measure intrinsic curvature, and their definitions (1.2) and (1.3) are nearly identical. Suitably interpreted, the field-strength tensor F is the curvature tensor R in a $(4+n)$-dimensional space-time of which n dimensions are associated with the gauge fiber:

$$R_{\mu\nu}{}^a{}_b = F_{\mu\nu}{}^a{}_b \quad \text{where} \quad \begin{array}{ccc} \mu,\nu & : & 1\ldots 4 \\ a,b & : & 5\ldots 4+n \end{array}. \tag{5.3}$$

Here the range of each index is restricted to the appropriate vector bundle dimensions, and the metric is formed by joining block diagonally the space-time metric with the gauge-fiber metric. Also we use only the coordinates of the base manifold to perform integrations and differentiations; we call such coordinates *traversable*. By construction, the origins of the tangent bundle T_xM and of the gauge vector bundle $F|_x$ are fixed. The only manifold whose coordinates can be traversed is the base space-time manifold.

The Ricci tensor and the curvature scalar cannot be formed from $R_{\mu\nu}{}^a{}_b$ under these conditions. In effect, because the metric components $g^{\mu a}$ vanish, both the Ricci tensor $R_{\mu\nu ab}\, g^{\nu a}$ and the curvature scalar $R_{\mu\nu ab}\, g^{\nu a} g^{\mu b}$ also vanish. To include in the action a measure of the gauge-fiber curvature, we must use higher-order terms that do not appear in general relativity. The term

$$S_G = \int d^4x\sqrt{-g}\; \frac{1}{2\,q^2} F_{\mu\nu}{}^a{}_b F^{\mu\nu b}{}_a. \tag{5.4}$$

is the first gauge-invariant, Lorentz-invariant, higher-order measure of curvature in the gauge fiber that does not vanish due to symmetries or the block diagonal metric. This is a hint that the term $R_{\mu\nu\sigma\lambda}R^{\mu\nu\sigma\lambda}$ may be a correction to the action of general relativity.

The action of Riemannian gauge theory, constructed as closely as possible to that of general relativity, is the traditional gauge-theory action $S = S_M + S_G$ written in terms of basis-independent expressions.

6. Implications of Riemannian Gauge Theory

6.1 Non-compact Gauge Groups

In section 4, we mentioned that Narasimhan and Ramanan have shown that the choice of an orthonormal basis $\langle e_a, e_b \rangle = \delta_{\bar{a}b}$ on the gauge fiber $F|_x$ leads to an $SO(n)$ or $U(n)$ gauge theory. In Eqs. (5.1–5.4), we wrote the action of a general Riemannian gauge theory in terms of basis-independent quantities. A gauge transformation is a change in the choice of basis vectors used to describe vectors in the gauge fiber. Since the action of Riemannian gauge theory is independent of the choice of basis, we can choose any linearly independent basis for the gauge fibers $F = \mathbf{R}^n$ and $F = \mathbf{C}^n$.

When the gauge basis vectors are allowed to be an arbitrary linearly independent set, then the symmetry group of the fiber is $GL(n, \mathbf{R})$ or $GL(n, \mathbf{C})$, and not just $SO(n)$ or $U(n)$. The action of Riemannian gauge theory and the quantities that follow from it are invariant under $GL(n, \mathbf{R})$ or $GL(n, \mathbf{C})$ gauge transformations.

In traditional gauge theory, one includes in the action every term that is renormalizable and gauge invariant In a traditional gauge theory [25, 26] of the non-compact group $GL(n, \mathbf{C})$, the term

$$S_g = m^2 \int d^4x \sqrt{-g}\, (D_\mu g)_{a\bar{b}}\; g^{\bar{b}c}\, (D^\mu g)_{c\bar{d}}\, g^{\bar{d}a} \tag{6.1}$$

occurs because it is invariant; it gives a mass to the gauge bosons associated with the non-compact generators. The physical content of this theory changes, however, when it is interpreted as a Riemannian gauge theory. In this case, as we now show, the term (6.1) vanishes; all the gauge bosons are massless; and the ones associated with the non-compact generators are merely gauge artifacts. The reason is that the covariant derivative of the gauge fiber metric [25, 26]

$$(D_\mu g)_{\bar{a}b} = \partial_\mu g_{\bar{a}b} - i\, (A_\mu)^{\bar{c}}{}_{\bar{a}} g_{\bar{c}b} + i\, g_{\bar{a}c}\, (A_\mu)^c{}_b \tag{6.2}$$

vanishes in Riemannian gauge theory. For if we differentiate the definition (4.6) of the metric

$$\partial_\mu g_{\bar{a}b} = (\partial_\mu \tilde{e}_{\bar{a}}{}^{\bar{j}})\, \tilde{e}_{b\bar{j}} + \tilde{e}_{\bar{a}}{}^{\bar{j}}\, (\partial_\mu \tilde{e}_{b\bar{j}}), \tag{6.3}$$

and use the metric to raise and lower indices, *e.g.*, $e^c{}_j g_{c\bar{b}} \delta^{j\bar{k}} = e_{\bar{b}}{}^{\bar{k}}$, then with an appropriate complex conjugation, we find

$$\partial_\mu g_{\bar{a}b} = (\partial_\mu \tilde{e}_{\bar{a}}{}^{\bar{j}})\, \tilde{e}^{\bar{c}}{}_{\bar{j}}\, g_{\bar{c}b} + g_{c\bar{a}}\, \tilde{e}^c{}_j\, (\partial_\mu \tilde{e}_b{}^j). \tag{6.4}$$

Using the definition (4.7) of the gauge field, we see that the covariant derivative of the metric vanishes, $(D_\mu g)_{\bar{a}b} = 0$, and so the term (6.1) does not contribute to the action of the Riemannian gauge theory.

In summary, the action of Riemannian gauge theory Eqs. (5.1 – 5.4) is basis independent. The choice of non-orthonormal basis gives a larger $GL(n, \mathbf{R})$ or $GL(n, \mathbf{C})$ symmetry without adding the term (6.1) to the action.

Freedman, Haagensen, Johnson, Latorre, and Lam [27, 28, 29] have studied the Riemannian gauge theory of the real gauge groups $GL(3, \mathbf{R})$ and $SO(3)$. They identified directions on the gauge fiber $F = \mathbf{R}^3$ with spatial directions on the base manifold. This identification is possible because the gauge fiber metric satisfies $(D_\mu g)_{ab} = 0$. The resulting curved manifold represents the geometry of the gauge fiber.

The covariant derivative of the gauge fiber metric vanishes because the gauge field and the metric are defined in terms of basis vectors. This result is reminiscent of general relativity. There the covariant derivative of the space-time metric vanishes because the connection coefficients and the metric are defined in terms of tangent basis vectors.

6.2 Uniqueness of the Coupling Constant

A traditional gauge theory of a compact simple group has a unique coupling constant and discrete charges (eigenvalues). However, a traditional gauge theory of an abelian group, which has no mathematical distinction between the coupling constant and the charges, can couple to each field with a different coefficient in the covariant derivative. So there is problem of charge quantization in the traditional gauge theory of an abelian group.

Riemannian gauge theory offers a solution to this problem of traditional gauge theory. In the definition (4.3) of the covariant derivative, $D_\sigma(\phi) = e_a\,(\delta^a_b \partial_\sigma + e^a(\partial_\sigma e_b))\,\phi^b$, the relation $(A_\mu)^b{}_a = i\,e^b\,(\partial_\mu e_a)$ between the gauge field and the basis vectors has no adjustable parameter even in the abelian case $(A_\mu)^1{}_1 = i\,e^1\,(\partial_\mu e_1)$. This lack of an adjustable parameter is expected in the covariant derivative of an $SU(n)$ gauge theory, but it is surprising in an abelian gauge theory.

In $U(1)$ gauge theory, this lack of an adjustable parameter leads to $U(1)$ charge quantization. Every matter-field vector $\phi^1\,e_1$ on the gauge fiber F couples with the same coefficient. The result is stronger than charge quantization, it is charge uniqueness. The uniqueness of the coupling in the covariant derivative arises because the matter fields are defined as vectors on a flat gauge vector bundle and because the gauge-covariant derivative is defined like the covariant derivative of Riemannian geometry. Just as we cannot choose the coefficient of $\Gamma^\lambda_{\mu\nu}$, so too we cannot choose the coefficient of A_μ.

A single adjustable parameter for a particular vector bundle occurs as the coefficient of the field strength term $q^{-2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$. This coefficient adjusts the energy cost of intrinsically curving the gauge fiber.

We have shown that all matter fields as vectors on a gauge vector bundle will couple in the covariant derivative with the same coupling constant. This unique coupling constant does not preclude positive and negative charges. Whenever one has two self-conjugate fields ϕ^1 and ϕ^2 of the same mass, one may form the complex field $\phi = (1/\sqrt{2})(\phi^1 + i\phi^2)$, which creates a particle and deletes its antiparticle; if the particle has charge q, then the antiparticle has charge $-q$. The particle and antiparticle correspond to the two solutions $D_t\phi = \pm\, i\,\omega\,\phi$. Matter-field vectors $\phi^a e_a$ with opposite charges rotate in opposite directions on the gauge fiber as time goes by. Localized matter-field vectors with opposite directions of rotation on a background curved gauge fiber of an electromagnetic field accelerate in opposite spatial directions.

In Riemannian gauge theory, the coupling in the covariant derivative is independent of the field for the same reason that in general relativity the gravitational acceleration is independent of the mass. The equivalence principle entails that the manifold is everywhere locally flat, and therefore that every small neighborhood can be identified with a flat tangent space. Just as the tangent vectors t_μ describing the tangent space determine the connection (3.3) of the covariant derivative by the relation $\Gamma^\alpha_{\sigma\mu} = t^\alpha(\partial_\sigma t_\mu)$ without an adjustable parameter, so too the basis vectors e_a describing the gauge fiber determine the connection (4.4) of the gauge-covariant derivative by the relation $(A_\mu)^a{}_b = ie^a(\partial_\mu e_b)$ without an adjustable parameter. The use of basis vectors to describe gauge fibers generalizes the equivalence principle to gauge theory.

6.3 Multiple $U(1)$ Charges

The multiple, independent $U(1)$ charges mentioned in section 1 can occur in Riemannian gauge theory, albeit unnaturally. Using the methods of section 4, we create two independent $U(1)$ gauge fields $-iB_\mu = b^1(\partial_\mu b_1)$ and $-iC_\mu = c^1(\partial_\mu c_1)$. The basis vectors b_1 and c_1 are the basis vectors of two *a priori* independent 1-complex-dimensional gauge vector bundles, $F^{(B)}$ and $F^{(C)}$. To get different charges related to the same gauge field, we need to constrain the intrinsic curvature of the two fibers $F^{(B)}$ and $F^{(C)}$ such that the connections are related by $B_\mu/q^B = C_\mu/q^C = A_\mu$. The geometry of two *a priori* independent gauge fibers has been constrained and expressed in terms of the variable A_μ. Matter fields on F^B couple with the covariant derivative $(\partial_\mu - iq^B A_\mu)$, and matter fields on F^C couple with the covariant derivative $(\partial_\mu - iq^C A_\mu)$. So by putting different matter fields on different fibers, we may give them different charges.

But if all electrons, muons, and taus are represented by vectors on the same gauge fiber, then they naturally have the same electric charge. Thus Riemannian gauge theory shifts the electric charge quantization problem. Instead of wondering why so many $U(1)$ matter fields couple with the same charge q_e, we ask: Why do the quark charges differ among themselves and from the charge of the electron? Why should matter fields lie on fibers with curvatures that are related in peculiar ways? Grandly unified theories provide a natural way of constraining the curvature of different fibers.

6.4 Grand Unification

The $SU(5)$ theory of Georgi and Glashow [5] demonstrates how the curvature of different one-complex-dimensional fibers may be constrained to produce different $U(1)$ charges in a theory of grand unification.

We create a 5-complex-dimensional vector bundle over space-time spanned by five orthonormal basis vectors e_a. We restrict $e^a(\partial_\mu e_a) = 0$ to make the group $SU(5)$ instead of $U(5)$. The hypercharge gauge field B_μ is identified with the gauge field proportional to the diagonal generator λ_{24}

$$-iB_\mu = -\frac{1}{3}e^1(\partial_\mu e_1) - \frac{1}{3}e^2(\partial_\mu e_2) - \frac{1}{3}e^3(\partial_\mu e_3) + \frac{1}{2}e^4(\partial_\mu e_4) + \frac{1}{2}e^5(\partial_\mu e_5). \qquad (6.5)$$

The matter-field covariant derivative is still $D_\mu \phi = e_b \left(\delta^a_b \partial_\mu + e^a(\partial_\mu e_b)\right) \phi^a$, but only $\frac{1}{3}$ of the term $e^1(\partial_\mu e_1)$ is identified as coupling to the hypercharge B_μ. This example shows how $SU(5)$ unification constrains the curvature of different one-complex-dimensional gauge fibers so as to give different $U(1)$ charges.

Since Riemannian gauge theory constrains the coupling in the covariant derivative, the group $U(n)$ may be used in theories of grand unification.

7. Conclusion

Gauge theories traditionally have been defined by transformation rules for fields under a symmetry group. All gauge-invariant, renormalizable terms are included in the action. The resulting gauge theories have many parallels with Riemannian geometry. In this paper, we have constructed a gauge theory based upon Riemannian geometry, which we have called Riemannian gauge theory.

Drawn from general relativity, the action of Riemannian gauge theory in a particular gauge is the action of traditional gauge theory. To measure the curvature of the gauge fiber, it is necessary to use a term that is quadratic in the curvature tensor. A Riemannian gauge theory with a $U(n)$ or $SO(n)$ gauge symmetry is automatically invariant under the larger gauge group $GL(n, \mathbf{C})$ or $GL(n, \mathbf{R})$; no extra terms are needed in the action.

Riemannian gauge theory offers a solution to the problem of the quantization of charge in abelian gauge theories. The basis vectors e_a describing the gauge fiber determine the connection (4.4) of the gauge-covariant derivative by the relation $(A_\mu)^a{}_b = ie^a(\partial_\mu e_b)$ without an adjustable parameter.

Riemannian gauge theory describes the four forces more symmetrically, enlarges the gauge group, and solves the problem of charge-quantization in abelian gauge theories.

Acknowledgments

We should like to thank Yang He, Jun Song, Paul Alsing, and Joshua Erlich for many hours of helpful discussions. One of us (M. S.) would like to thank David Cardimona for his supervision of this effort.

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